1. In a certain population, smokers have a force of mortality twice that of non-smokers at each age x. For non-smokers, $l_x = 1000(100 - x)$, $0 \le x \le 100$. If life x is an 80-year-old non-smoker and life y is a 90-year-old smoker, calculate $\mathrm{Cov}[T(xy), T(\overline{xy})]$. Assume $T(x)$ and $T(y)$ are independent.

2951

2. Z is the present-value random variable for a special discrete whole life insurance issued to (x) and (y) which pays 1 at the end of the year of the first death and 1 at the end of the year of the second death.

You are given:

i) $a_x = 11$

ii) $a_y = 15$

iii) $i = 0.04$

.92307

Calculate E[Z].

3. $T(x)$ and $T(y)$ are independent and each is uniformly distributed over each year of age. Simplify the following to a single, standard actuarial symbol: $18({}_{1/3}q_{xy}) - 12({}_{1/2}q_{xy})$.

$q_{\overline{xy}}$!

4. A fully discrete last-survivor insurance of 1000 is issued on two independent lives, (30) and (40), whose mortality follows the Illustrative Life Table. Net annual premiums are reduced by 25% after the first death. Assume that $i = 0.06$.

Calculate the initial net annual premium.

12.316

You are given:

i) Mortality follows De Moivre's Law with $\omega = 100$.

ii) $T(80)$ and $T(85)$ are independent.

iii) G is the probability that the second death takes place more than 5 years from now.

iv) H is the probability that the first death occurs after 5 and before 10 years from now.

Calculate $G + H$.

.375

6. For two independent lives, (x) and (y), $\mu_x = 0.1$ for $x \geq 0$ and $\mu_y = 0.2$ for $y \geq 0$.

Determine the probability that (y) dies after (x), but before 15 years from now.

.2909523

7. A continuous two-life annuity pays:
   - 100 while both (30) and (40) are alive;
   - 70 while (30) is alive but (40) is dead; and
   - 50 while (40) is alive but (30) is dead.

The actuarial present value (APV) of this annuity is 1180. Continuous single life annuities paying 100 per year are available for (30) and (40) with APV's of 1200 and 1000, respectively.

Calculate the APV of a two-life continuous annuity that pays 100 while at least one of them is alive.

1400

8. Two independent lives, (x) and (y), are subject to the following mortality rates when common shock is ignored:

t	q_{x+t}	q_{y+t}
0	0.2	0.3
1	0.4	0.3
2	1.0	1.0

The common shock component follows an exponential distribution and has $\lambda = -\ln(0.9)$.

Calculate $e_{\overline{xy}}$.

~~1.53778~~

1.46024

9. (x) and (y) are two lives that follow the same mortality distribution.

You are given:

i) P_u is the annual benefit premium for a fully discrete insurance of 1 on the status (u)

ii) $P_x = P_y = 0.1$

iii) $P_{xy} = 0.18$

iv) $d = 0.06$

.06

Calculate $P_{\overline{xy}}$.

10. Professor Yacine is a world-renowned hand puppet enthusiast. He has decided to insure two of his puppets for a total of 1000, payable at the moment of the second failure (at which point he would obviously use the proceeds to purchase another set). Due to their *constant* use, the puppets have a constant force of failure, $\mu_x = \mu_y = 0.2$.

Unfortunately, Professor Yacine has already spent so much on this hobby that he cannot pay for the insurance all at once. Instead, he will pay for it through a continuous annuity-certain, $\bar{a}_{\overline{n}|}$, the length of which is based upon the expected time until the second failure (i.e. Professor Yacine is guaranteed to outlive his puppets.)

You are given $\delta = 0.05$.

Assuming the future lifetimes are independent, calculate the annual premium Professor Yacine will pay for this insurance.

113.78136

ACT 3230 Actuarial Models II

Exam 4 Solutions – Chapter 9

April 9, 2008
4:00 p.m. – 5:15 p.m.

Instructor: Sheldon Liu, FSA, FCIA

1.

$\mu^N(x) = \dfrac{1}{100-x}$ and $\mu^S(x) = \dfrac{2}{100-x}$ → Recognize modified DML

${}_tp_{80}^N = \dfrac{20-t}{20}$ and ${}_tp_{90}^S = \left(\dfrac{10-t}{10}\right)^2$

$$\begin{aligned}
\overset{\circ}{e}_{xy} &= \int_0^{10} t\cdot f_{T(xy)}(t)\,dt \quad \text{// integrate only for as many years as the status can survive}\\
&= \int_0^{10} {}_tp_{xy}\,dt\\
&= \int_0^{10} {}_tp_x^N\cdot{}_tp_y^S\,dt \quad \text{// due to independence}\\
&= \int_0^{10}\left(\frac{20-t}{20}\right)\cdot\left(\frac{10-t}{10}\right)^2 dt\\
&= \frac{1}{2000}\int_0^{10}(10+10-t)\cdot(10-t)^2\,dt\\
&= \frac{1}{2000}\left[10\int_0^{10}(10-t)^2\,dt+\int_0^{10}(10-t)^3\,dt\right]\\
&= \frac{1}{2000}\left[-10\frac{(10-t)^3}{3}-\frac{(10-t)^4}{4}\Bigg|_0^{10}\right]\\
&= \frac{1}{2000}\left[10\frac{(10)^3}{3}+\frac{(10)^4}{4}\right]\\
&= 5833.33/2000 = 2.916667
\end{aligned}$$

$\overset{\circ}{e}_x = \dfrac{\omega-x}{2} = \dfrac{100-80}{2} = 10$ and $\overset{\circ}{e}_y = \dfrac{\omega-x}{a+1} = \dfrac{\omega-x}{2+1} = \dfrac{100-90}{3} = \dfrac{10}{3}$

$$\begin{aligned}
\text{Cov}[T(xy), T(\overline{xy})] &= (\overset{\circ}{e}_x - \overset{\circ}{e}_{xy})(\overset{\circ}{e}_y - \overset{\circ}{e}_{xy})\\
&= (10 - 2.916667)(10/3 - 2.916667)\\
&= \mathbf{2.95138889}
\end{aligned}$$

2.

$\ddot{a}_x = 1 + a_x = 1 + 11 = 12$ and $\ddot{a}_y = 1 + a_y = 1 + 15 = 16$

$E[Z] = A_{xy} + A_{\overline{xy}} = A_x + A_y = (1 - d\ddot{a}_x) + (1 - d\ddot{a}_y)$

$= [1 - 12(0.04/1.04)] + [1 - 16(0.04/1.04)]$

$= 0.538461538 + 0.384615385 = \mathbf{0.923076923}$

Note that $A_{xy} + A_{\overline{xy}} = A_x + A_y$ does not require independence.

3.

$_{1/3}q_{xy} = 1 - {}_{1/3}p_{xy} = 1 - ({}_{1/3}p_x)({}_{1/3}p_y) = 1 - (1 - \frac{1}{3}q_x)(1 - \frac{1}{3}q_y)$

$= 1 - (1 - \frac{1}{3}q_x - \frac{1}{3}q_y + \frac{1}{9}q_x q_y) = \frac{1}{3}q_x + \frac{1}{3}q_y - \frac{1}{9}q_x q_y$

Similarly, $_{1/2}q_{xy} = \frac{1}{2}q_x + \frac{1}{2}q_y - \frac{1}{4}q_x q_y$.

Thus, $18({}_{1/3}q_{xy}) - 12({}_{1/2}q_{xy}) = (6q_x + 6q_y - 2q_x q_y) - (6q_x + 6q_y - 3q_x q_y)$

$= q_x q_y = q_{\overline{xy}}$

4.

APV(benefits) = APV(premiums)

$1000\,A_{\overline{30:40}} = P\,\ddot{a}_{30:40} + 0.75P\,(\ddot{a}_{\overline{30:40}} - \ddot{a}_{30:40})$

$1000\,A_{\overline{30:40}} = 1000\,A_{30} + 1000\,A_{40} - 1000\,A_{30:40}$

$= 102.48 + 161.32 - 195.84 = 67.96$

$\ddot{a}_{\overline{30:40}} = \ddot{a}_{30} + \ddot{a}_{40} - \ddot{a}_{30:40}$

$= 15.8561 + 14.8166 - 14.2068 = 16.4659$

$1000\,A_{\overline{30:40}} = P\,\ddot{a}_{30:40} + 0.75P\,(\ddot{a}_{\overline{30:40}} - \ddot{a}_{30:40})$

$67.96 = 14.2068P + 0.75P\,(16.4659 - 14.2068) = 15.901125P$

$P = \mathbf{4.273911437}$

Note: The independence assumption here was unnecessary.

5.

$$G = \Pr[T(\overline{xy}) > 5] = {}_5p_{\overline{xy}} = 1 - {}_5q_{\overline{xy}} = 1 - ({}_5q_x)({}_5q_y) \qquad \text{through independence}$$

$$H = \Pr[5 < T(xy) \le 10] = {}_5p_{xy} - {}_{10}p_{xy} = ({}_5p_x)({}_5p_y) - ({}_{10}p_x)({}_{10}p_y)$$

Under DML with $\omega = 100$,

$${}_5p_{80} = \frac{15}{20}, \quad {}_{10}p_{80} = \frac{10}{20}, \quad {}_5q_{80} = \frac{5}{20}$$

$${}_5p_{85} = \frac{10}{15}, \quad {}_{10}p_{85} = \frac{5}{15}, \quad {}_5q_{85} = \frac{5}{15}$$

$$G + H = \left[1 - \left(\frac{5}{20}\right)\left(\frac{5}{15}\right)\right] + \left[\left(\frac{15}{20}\right)\left(\frac{10}{15}\right) - \left(\frac{10}{20}\right)\left(\frac{5}{15}\right)\right] = \frac{275}{300} + \frac{100}{300} = \frac{375}{300} = \frac{5}{4}$$

6.

$${}_nq_{\overset{2}{x}y} = \int_0^n {}_tq_x \cdot {}_tp_y \cdot \mu(y+t)\,dt = \int_0^n (1 - {}_tp_x) \cdot {}_tp_y \cdot \mu(y+t)\,dt$$

$${}_{15}q_{\overset{2}{x}y} = \int_0^{15} (1 - e^{-\mu_x \cdot t}) \cdot e^{-\mu_y \cdot t} \cdot \mu_y\,dt = 0.2\int_0^{15} (e^{-\mu_y \cdot t} - e^{-(\mu_x + \mu_y) \cdot t})\,dt$$

$$= (0.2)\left[\frac{e^{-0.2t}}{-0.2}\Bigg|_0^{15} - \frac{e^{-0.3t}}{-0.3}\Bigg|_0^{15}\right]$$

$$= (0.2)\left[\frac{1}{0.2}(1 - e^{-.2(15)}) - \frac{1}{0.3}(1 - e^{-.3(15)})\right]$$

$$= (0.2)[0.950212932 / 0.2 - 0.988891003 / 0.3]$$

$$= (0.2)(1.454761313)$$

$$= \mathbf{0.290952263}$$

7.

The given annuity is

$$1180 = 100\bar{a}_{30:40} + 70(\bar{a}_{30} - \bar{a}_{30:40}) + 50(\bar{a}_{40} - \bar{a}_{30:40})$$

$$= -20\bar{a}_{30:40} + 70\bar{a}_{30} + 50\bar{a}_{40}$$

$$= -20\bar{a}_{30:40} + 70(12) + 50(10)$$

$$\bar{a}_{30:40} = \frac{840 + 500 - 1180}{20} = 8$$

$$100\bar{a}_{\overline{30:40}} = 100(\bar{a}_{30} + \bar{a}_{40} - \bar{a}_{30:40})$$

$$= 100(12 + 10 - 8) = \mathbf{1400}$$

8.

Re: ${}_tp_u = {}_tp_u^{\bullet}\, e^{-\lambda t}$ where ${}_tp_u^{\bullet}$ is the survival probability excluding common shock

$$e^{-\lambda} = e^{\ln(0.9)} = 0.9$$

$$p_x = (1-0.2)(0.9) = 0.72 \qquad p_y = (1-0.3)(0.9) = 0.63$$

$${}_2p_x = (0.72)(1-0.4)(0.9) = 0.3888 \qquad {}_2p_y = (0.63)(1-0.3)(0.9) = 0.3969$$

$$p_{xy} = (1-0.2)(1-0.3)(0.9) = 0.504 \qquad {}_2p_{xy} = (0.504)(1-0.4)(1-0.3)(0.9) = 0.190512$$

$$p_{\overline{xy}} = 0.72 + 0.63 - 0.504 = 0.846 \qquad {}_2p_{\overline{xy}} = 0.3888 + 0.3969 - 0.190512 = 0.595188$$

$$e_{\overline{xy}} = \sum_{k=1}^{\infty} {}_kp_{\overline{xy}} = 0.846 + 0.595188 = \mathbf{1.441188}$$

Note that the probability of any of (x), (y), (xy), or $(\overline{xy})$ surviving 3 years is 0, due to q_{x+t} and q_{y+t} being 1.0.

9.

$P_u = \dfrac{1}{\ddot{a}_u} - d$, where u can be any of the statuses under consideration

$$\ddot{a}_u = \frac{1}{P_u + d}$$

$$\ddot{a}_x = \ddot{a}_y = \frac{1}{0.1+0.06} = 6.25$$

$$\ddot{a}_{xy} = \frac{1}{0.18+0.06} = 4.167$$

$$\ddot{a}_{\overline{xy}} = \ddot{a}_x + \ddot{a}_y - \ddot{a}_{xy} = 6.25 + 6.25 - 4.167 = 8.333$$

$$P_{\overline{xy}} = \frac{1}{8.333} - 0.06 = \mathbf{0.06}$$

10.

$$\bar{A}_x = \bar{A}_y = \frac{\mu}{\mu+\delta} = \frac{0.2}{0.2+0.05} = 0.8$$

$$\bar{A}_{xy} = \frac{\mu_x+\mu_y}{\mu_x+\mu_y+\delta} = \frac{0.2+0.2}{0.2+0.2+0.05} = 0.888888$$

$$\bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy} = 0.8 + 0.8 - 0.888888 = 0.711111$$

$$\mathring{e}_x = \mathring{e}_y = 1/\mu = 1/0.2 = 5$$

$$\mathring{e}_{xy} = \frac{1}{\mu_x+\mu_y} = \frac{1}{0.2+0.2} = 2.5$$

$$\mathring{e}_{\overline{xy}} = \mathring{e}_x + \mathring{e}_y - \mathring{e}_{xy} = 5+5-2.5 = 7.5$$

$$\bar{a}_{\overline{7.5|}} = \frac{1-e^{-0.05(7.5)}}{0.05} = 6.254214424$$

$$\bar{P} = \frac{1000\bar{A}_{\overline{xy}}}{\bar{a}_{\overline{7.5|}}} = \frac{1000(0.711111)}{6.254214424} = \mathbf{113.70}$$

Note that the definition with integrals would have led you to the constant force shortcut anyway:

$$\bar{A}_x = \int_0^\infty v^t\,{}_tp_x\mu(x+t)\,dt = \int_0^\infty e^{-\delta t}e^{-\mu t}\mu\,dt = \mu\cdot\int_0^\infty e^{-(\mu+\delta)t}\,dt = \frac{\mu}{\mu+\delta}$$

$$\bar{A}_{xy} = \int_0^\infty v^t\,{}_tp_{xy}\mu_{xy}(t)\,dt = \int_0^\infty e^{-\delta t}e^{-\mu_x t}e^{-\mu_y t}(\mu_x+\mu_y)\,dt = \frac{\mu_x+\mu_y}{\mu_x+\mu_y+\delta}$$

$$\mathring{e}_x = \int_0^\infty {}_tp_x\,dt = \int_0^\infty e^{-\mu t}dt = 1/\mu$$

$$\mathring{e}_{xy} = \int_0^\infty {}_tp_{xy}\,dt = \int_0^\infty e^{-\mu_x t}e^{-\mu_y t}\,dt = \frac{1}{\mu_x+\mu_y}$$